OPTIMAL CONTROL APPLIED TO ECONOMIC STQ STABILIZATION

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PART I

Since the article of Sengupta (1970), the optimal control approach to economic stabilization theory has envariably dealt with what is called fixed-time, free endpoint problems. In this type of optimization problems the control time, T, is fixed a priori and the final state of the economy is assumed to be free. The disadvantage of this approach is that at the exogenously fixed final time, T, the state of the economy might very well be far from the equilibrium state, in either direction. In which case we will have an under shooting or an over shooting Problem the purpose of this paper is to propose a different use of optimal control in stabilization theory, by solving an inverted problem, that can be called a fixed endpoint, free-time problem. Here the final time, T, is assumed to be free, to be determined endogenously by the optimization process. Instead, the final state of the economy is given a priori as equal to a given disired level. The concern here is to make sure that, by the end of the stabilization process the economic system reaches a given, a priori, target. In addition, to the old approach, here the optimization will be subject to, a control constraint, among other things. In the example to be studied, the goal is to find the time-path, and the total amount of, government expenditure necessary to transfer the economic system from a non-desired initial state to a desired final state.

1- THE MODEL USED

For an illustration purpose only, we shall use the following dynamic disequilibrium, multiplier accelerator model :

$$Z(t) = C(t) + I(t) + G(t) \qquad (1)$$

$$Y(t) = \frac{1}{F} \{ Y(t) - Z(t) \} \qquad (2)$$

Where the consumption $C(t)$ is given by

$C(t) = (1 - s)\ Y(t)$, s being the propensity to save, and the investment $I(t)$ is defined by :

$I(t) = vy(t)$, where v is the acceleration coefficient

$G(t)$ is an official demand, used as a control variable to stabilize the economy, according to the dynamic adjustment mechanism between aggregate demand and aggregate supply, as given by equation (2),

Where F is the time constant of the production lag, and $Y(t) = \frac{dy\ (t)}{dt}$

$Z(t)$ is, the aggregate demand measured from the initial equilibrium value, and $Y(t)$ is the aggregate production (indentified with national income) measured from the initial equilibrium level.

From above we got :

$$Z = (1 - s)\ Y + vy + G \qquad (3)$$

The variable t has been dropped for convenience, i. e., for example Y stands for Y(t).

Now some transformation on the variable G is needed in order to be able to apply optimal control theory to the problem under study. G will be the only control variable and it will be assumed that

$$\underline{G} \leqslant G \leqslant \bar{G}$$

where $\underline{G}$ is a lower level and $\bar{G}$ an upper level for government expenditure. Both $\underline{G}$ and $\bar{G}$ are constant.

Define g(t) by :

$$g(t) = 2\frac{G - \frac{1}{2}(\underline{G} + \bar{G})}{\bar{G} - \underline{G}} \qquad (4)$$

Then

$$/\ g(t)\ / \leqslant 1 \qquad (5)$$

We then get from above :

$$\dot{Y}_t = -ay + bg + c \qquad (6)$$

Where

$$a = \frac{s}{F - v} \qquad (7)$$

$$b = \frac{\bar{G} - \underline{G}}{F - v} \qquad (8)$$

$$c = \frac{\bar{G} + \underline{G}}{F - v} \qquad (9)$$

A change of origin $y \rightarrow y = y - \frac{c}{a}$ is now made so that equation (6) becomes

$$y = -ay' + bg \qquad (10)$$

In the text of the paper we use y to mean this y.

Equation (6) is a fundamental equation as it represents the dynamic of the economic system. As it is well known its solution is given by (see ref. 1) :

$$y(t) = y_o e^{-at} + e^{-at}\int_0^t e^{ai} \left[bg(i) \right] di \qquad (10)(bis)$$

where i is a (dummy) variable of integration.

II - THE PROBLEM

a/ Find the control g(t) that would transfer a nonzero initial state $y(O) = y_O$ at time $t = O$, to a zero final state $y(t) = O$ at time $t = T$, such that the following performance functional:

$$I_2 = \frac{1}{2} \int_0^T g^2 (t)\, dt \qquad (11)$$

is minimized, subject to :

$$y = - ay + bg \qquad (12)$$

with

$y(O) = y_O$

$y(T) = O$

T is free

and the control constraint,

$$|g| \leq 1 \qquad (13)$$

(we assume that $y(O) = y$ is transferable to $y(T) = O$

First let us form the hamiltonian function :

$$H = \frac{1}{2}g^2 + P\,(- ay + bg) \qquad (14)$$

b. Necessary conditions

b.1 The minimum principle gives :

$$H|y^*, g^*, p^*, t| \leq H|y^*, g\, p^*, t| \qquad (15)$$

$$\frac{1}{2} g^{*2} + p^*bg^* - p^*ay^*, - \frac{1}{2}g^{2*} + p^*bg - p^*ay^*, \qquad (17)$$

b. 2 The cononical system is given by :

$$H^*_p = \dot{y}^* = - ay^* + bg^* \qquad (18)$$

$$-H^*_y = \dot{p}^* = ap^* \qquad (19)$$

with

$y(O) = y_O$

and

$y(T) = O$

b. 3 Since the hamiltoniam function, as given bey equation (14) does not does not depend explicitly on time and the final time T is free, we have again, the following additional necessary conditions :

The hamiltonian function is equal to zero on the optimal trajectory, during all the time-interval (o,t), that is :

$$H(y^*, g^*, p^*) = 0 \qquad (20)$$

Then

$$\tfrac{1}{2}g^{*2} + p^*(-ay^* + bg^*) = 0 \qquad (21)$$

C. The possible optimal policies :

From equation (17) and for $|g(t)|$ 1, we will have an interior solution for the minimum of the hamiltonian function.

In this case the boundaries of the control constraint.

$$|g| \leq 1$$

do not affect the solution. the necessary and sufficient conditions to minimize the hamiltonian function (14), are then :

$$\frac{\partial H}{\partial g^*} = g^* + p^*b = 0 \qquad (22)$$

$$\frac{\partial^2 H}{\partial g^{*2}} = 1 > 0 \qquad (23)$$

Equation (23) assures us that the optimal control $g^*(t)$ will minimize the hamiltonian function (14).
Condition (22) gives us the optimal control as :

$$g^* = -p^*b \qquad (24)$$

when

$$|g| < 1$$

which amounts to :

$$g^* = -p^*b \quad \text{if } |p^*b| < 1 \qquad (25)$$

what happens then when

$$|p^*b| \geqslant 1 \qquad (26)$$

in this case and taking into account the control constraint.

$|g| \leqslant 1$ which implies $|g^*| \leqslant 1$

the optimal stabilization policy is :

$$g^* = \begin{cases} 1 & \text{if } p^*b \leqslant -1 \\ -1 & \text{if } p^*b \geqslant 1 \end{cases} \qquad (27)$$

in order for the hamiltonian to be minimum.

combining equations (25) and (27), the minimum principle shows that the optimal government expenditure $g^*(t)$ is such that :

$$g^* = \begin{cases} -1 & \text{if } p^*b \geqslant 1 \\ -p^*b & \text{if } |p^*b| < 1 \\ 1 & \text{if } p^*b \leqslant -1 \end{cases} \quad \text{for all } t \qquad (28)$$

Or

$$g^* = -\text{sat}(p^*b) \qquad (29)$$

where we define the saturation (sat)

$$\text{Sat } F = \begin{cases} F & |F| \leqslant 1 \\ \text{Sgn}F & |F| \geqslant 1 \end{cases} \qquad (30)$$

To be continued